Newton School of Technology

NSAT

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NSAT Exam Guidelines

Exam Protocol

- Arrive at least 15 minutes early (by 2:45 PM) for exam setup and proctoring checks.
- All three 3 sections are mandatory and include a sectional cutoff that determines passing eligibility
- The exam will be proctored. Unfair means will lead to permanent disqualification.
- For MCQ-type Questions, Positive marking will be (+4) for correct answers, while negative marking (-1) is applicable for incorrect ones.
- There is no negative marking in coding section.

Joining Details & Device Restrictions

- Use a laptop/PC with screen-sharing and microphone access
- Dual-camera setup required:
  - Primary camera (webcam) facing your face
  - Secondary camera (mobile/external) showing your workspace
- Mobile phones are only allowed as a secondary camera (not for taking the test)
- Calculators are prohibited; use pen and paper for rough work
- Access the test portal at https://my.newtonschool.co/nsat/timeline
- Accept the calendar invite sent to you on test day as a reminder
- Maintain a stable internet connection to prevent interruptions during NSAT
- Keep both cameras ON throughout the test
- Position cameras properly:
  - Primary camera: face clearly visible
  - Secondary camera: desk, hands, and screen visible
- Ensure the secondary camera is placed steadily (not handheld)

Preparation

- Visit NSAT homepage to check compatibility before the exam.
- Use Google Chrome for optimal performance.
- Bring a pen and paper for rough work during the exam.
- Attempt a mock test to understand the exam pattern, for practice, and to confirm your PC compatibility.

Environment Considerations

- Sit in a quiet environment with no background noise to minimize distractions and avoid disqualification.
- Ensure the lighting in the room is appropriate for clear visibility and comfortable reading and writing.

PERMUTATION & COMBINATION

Permutations: In mathematics, a permutation is the orderly placement of every element in a set in a specific sequence. Permutations are important in many areas of mathematics, especially when examining different arrangements of finite sets.

$$^{n}P_{r}=\frac{n!}{(n-r)!}$$

1. Permutation with repetition: When each object can be repeated an arbitrary number of times, the total number of permutations of *n* distinct objects taken *r* at a time is *n*
2. Permutation of similar objects: When *n* objects are taken and *p* are like objects of one kind, *q* are like objects of a second kind, and *r* are like objects of a third kind, then the permutation of n objects taken all at a time is $\frac{n!}{p!q!r!}$.
3. Permutation under Restriction: When *m* specified objects always arrive together, the number of permutations of *n* distinct objects, considered all at once, is $m!(n-m+1)!$

Combinations: Selecting pieces from a collection without taking the order into account is known as a combination. The formula for combinations is $^{n}C_{r}=\frac{n!}{(n-r)!r!}$.

Must Know Formulae:

1. Circular arrangement: $(n-1)!$
2. $^{n}C_{r}={}^{n}C_{n-r}$
3. $^{n}C_{r}+{}^{n}C_{r-1}={}^{n+1}C_{r}$
4. $\frac{^{n}C_{r}}{^{n}C_{r-1}}=\frac{n-r+1}{r}$
5. When (m + n) distinct objects are split into two unequal groups with *m* and *n* objects, respectively, the number of possible arrangements is $^{m+n}C_{n}=\frac{(m+n)!}{m!n!}$
6. If p specific objects need to be excluded, the number of ways to select r objects from n various objects is equal to $^{n-p}C_{r}$
7. The total number of combinations if, among *(p + q + r + s)* objects, *p* is of one kind, *q* is of a second kind, *r* is of the third kind, and *s* are different, is: $(p+1)(q+1)(r+1)2^{s}-1$.

Concepts to be Applied:

The 'Beggar's Method': Sometimes referred to as the 'Stars and Bars' approach, is a combinatorial methodology that determines how many ways there are to freely distribute identical goods. The formula for the Beggar's method is ${}^{r+n-1}C_{n-1}$.

Example: Distribute 5 identical coins among 3 beggars, such that each beggar gets at least 1 coin.

Solution: The beggars must get at least 1 coin so first distribute 3 coins amongst the beggars. Now we are left with $r=5-3=2$ coins.

Now the number of ways to distribute the remaining $r=2$ coins among $n=3$ beggar's can be found using the star and bar approach as follows:

$${}^{3+2-1}C_{3-1}={}^{4}C_{2}=\frac{4!}{2!\times2!}=\frac{4\times3\times2!}{2!\times2!}=3\times2=6$$

The number of possible ways is 6.

Derangement: A derangement is a permutation of a set's elements in which no element stays in the same location. The formula for derangement is, $D_n=n!\sum_{k=0}^{n}\frac{(-1)^k}{k!}$

Example: There are 4 marbles of different colors and 4 glasses of the same colors. Find the number of ways in which the marbles can be placed one in each glass such that a marble does not go to a glass bearing the same color.

Solution: The number of possible ways can be found using the derangement concept as follows.

For $n=4$ the number of derangements is = $4!\left\{1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right\}=12-4+1=9$

The number of such arrangements is 9.

Standard Results:

1. On a plane with $m(<n)$ collinear, the number of completely distinct straight lines created by linking n points is ${}^{n}C_2-{}^{m}C_2+1$
2. On a plane with $m(<n)$ collinear points, the total number of triangles created by linking n points is ${}^{n}C_3-{}^{m}C_3$
3. If a family of n parallel lines intersect m parallel lines in a plane, the total number of parallelograms created are ${}^{m}C_2\times{}^{n}C_2$
4. If a circle has n points then:

The number of possible straight lines between these points is ${}^{n}C_2$

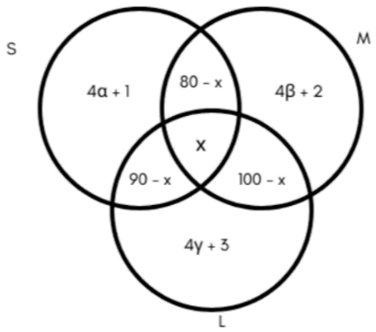
The number of possible triangles is ${}^{n}C_3$

The number of possible quadrilaterals is ${}^{n}C_4$

Example 1: In a class of 400 students 80 took Science(S) and Mathematics(M) both, 90 took Language(L) and Science(S) both, 100 took Mathematics(M) and Language(L). The number of students who took all three is a multiple of 10, while the number of students who took S only, M only and L only leaves remainders 1,2 and 3 respectively when divided by 4, then what is the total number of possible combinations of influence?

Solution:

Let the number of students who took all three be x. Make a Venn diagram with



suitable variables. The values of α, β and γ will be whole numbers.

The maximum number of students in the class is 400. So,

$$4(\alpha+\beta+\gamma)+6+270-2x \le 400$$
$$4(\alpha+\beta+\gamma)-2x \le 124$$
$$2(\alpha+\beta+\gamma)-x \le 62$$

The value X is a multiple of 10. The values that X can take are $[0,10,20,....80]$.

The possible cases for different X values will be:

$$x=0:\alpha+\beta+\gamma \le 31$$

By beggar's method the possible solutions ${}^{31+3}C_3 = {}^{34}C_3$

$$x=10:\alpha+\beta+\gamma \le 36$$

By beggar's method the possible solutions ${}^{36+3}C_3 = {}^{39}C_3$

The total number of solutions from $x=0$ to $x=80$ will be:

$$\text{Total solutions} = {}^{34}C_3 + {}^{39}C_3 + ... + {}^{74}C_3$$
$$= \sum_{i=1}^{9} {}^{5r+29}C_3$$

The total number of possible combinations of influence is $\sum_{i=1}^{9} {}^{5r+29}C_3$.

Example 2: If p,q,r are prime numbers and α,β,γ are positive integers such that the least common multiple (L.C.M.) of α,β,γ is p^3q^3r and the highest common factor (H.C.F) is pqr, then what will be the number of possible triplets (α,β,γ)?

Solution:

The $LCM(\alpha,\beta,\gamma)=p^3q^3r^1$ and $HCF(\alpha,\beta,\gamma)=p^1q^1r^1$ have the power of r^1 common.

The triplets will be of the form $\alpha=p^{x_1}q^{y_1}r^1, \beta=p^{x_2}q^{y_2}r^1, \gamma=p^{x_3}q^{y_3}r^1$.

The exponent of p can vary such that it must contain at least a maximum exponent $\max(x_1,x_2,x_3)=3$ and minimum exponent $\min(x_1,x_2,x_3)=1$.

Possible Cases for the power of p are:

Case 1: $(1,1,3)$: The number of ways is $\frac{3!}{2!}=3$

Case 2: $(1,2,3)$: The number of ways in which powers can be arranged is $3!=6$

Case 3: $(1,3,3)$: The number of ways in which powers can be arranged is $\frac{3!}{2!}=3$

Total number of ways is $3+6+3=12$

Similarly, for q the possibilities are $\max(y_1,y_2,y_3)=3$ and $\min(y_1,y_2,y_3)=1$.

Similar case will be formed for q and 12 different ways are possible to get the exponents.

The number of possible triplets will be:

Number of ways $=12\times12\times1=144$

The number of possible triplets is 144

EXERCISE

1. x_1, x_2, x_3, x_4 are different prime divisors of number $N = 21000$. What is the total number of quadruples (x_1, x_2, x_3, x_4) for which $(\log_{2020}(\log_2 x_1))\cdot(\log_{2021}(\log_3 x_2))\cdot(\log_{2022}(\log_5 x_3))\cdot(\log_{2023}(\log_7 x_4)) \neq 0$?

2. Let n be the number of ways in which 5 males and 5 females can stand in a queue in such an arrangement that all the females stand together. Let q be the number of ways in which 5 males and 5 females can stand in a queue such that precisely 4 females stand together in the queue. Find $\frac{q}{n}$.

SOLUTIONS

1. The prime factorization of the number is $21000 = 2^3 \cdot 3 \cdot 5^3 \cdot 7$

   For the expression to be not equal to zero.

$$(\log_{2020}(\log_2 x_1)) \neq 0$$
$$\log_2 x_1 \neq 1$$
$$x_1 \neq 2$$

   Similarly on solving the remaining expressions we get the condition $x_1 \neq 2, x_2 \neq 3, x_3 \neq 5, x_4 \neq 7$.

   Apply the derangement formula for n = 4 we get:

$$D_4 = 4!\left(\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right)$$
$$= 9$$

   The total number of quadruples is 9.

2. Case 1: When 5 females stand together $(F_1, F_2, F_3, F_4, F_5), M_1, M_2, M_3, M_4, M_5$

   Consider the 5 females as one unit so the number of ways in these bounded females and the males can be arranged is $n = 6! \times 5!$

   Case 2: When exactly 4females stand together.
   $(F_1, F_2, F_3, F_4), M_1, M_2, M_3, M_4, M_5, F_5$

   The number of ways to select and arrange exactly 4 females is ${}^5C_4 \times 4!$

   The case $(F_1, F_2, F_3, F_4), F_5, M_1, M_2, M_3, M_4, M_5$ should not be considered. So, first only consider the males and arrange them, ${}^5C_4 \times 4! \times 5!$

The arrangement of males in the line will create 6 gaps so arrange the females in the gaps. The number of final arrangements will be $q = {}^{5}C_4 \times 4! \times 5! \times {}^{6}C_2 \times 2!$

Take the ratio to get:

$$\frac{q}{n} = \frac{{}^{5}C_4 \times 4! \times 5! \times {}^{6}C_2 \times 2!}{6! \times 5!} = \frac{5 \times 4! \times 5! \times 15 \times 2}{6! \times 5!} = \frac{5 \times 15 \times 2}{6 \times 5} = 5$$

The value $\frac{q}{n} = 5$

PROBABILITY AND STATISTICS

Probability is a branch of mathematics that quantifies the likelihood of an event occurring. It ranges from 0 (an impossible event) to 1 (a certain event). It helps in making predictions based on known information.

Important Concepts

1. **Sample Space (S):** The set of all possible outcomes.
2. **Types of Events:**
   - **Independent Events:** Events where the occurrence of one does not affect the occurrence of another.
   - **Dependent Events:** Events where the occurrence of one affects the occurrence of another.
   - **Disjoint Events**: Events that have no common outcomes. If A and B are disjoint, then $P(A \cap B) = 0$.
   - **Mutually Exclusive Events:** Events that cannot happen simultaneously.
   - **Complementary Events:** If event A occurs, its complement A' does not occur, and vice versa.
   - **Equally Likely Events:** All outcomes have the same probability.
   - **Exhaustive Events**: A set of events that covers all possible outcomes.
3. **Conditional Probability:**
   - The probability of an event occurring given that another event has already occurred.
4. **Probability with and without Replacement:**
   - With Replacement: Events remain independent.
   - Without Replacement: Events become dependent.

Important Formulae

1. **Probability of an Event:** $P(A)=\frac{n(A)}{n(S)}$.

2. For any event A, $0\le P(A)\le 1$.

3. For a sample space S, $P(S)=1$.

4. **Addition Rule:**

   - For 2 events: $P(A\cup B)=P(A)+P(B)-P(A\cap B)$.

   - For 3 events:

     $P(A\cup B\cup C)=P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)-P(B\cap C)+P(A\cap B\cap C)$

5. **Multiplication Rule:**

   - Independent Events: $P(A\cap B)=P(A)P(B)$

   - Dependent Events: $P(A\cap B)=P(A)P(B|A)$

6. **Bayes' Theorem:**

   - For 2 events: $P(A|B)=\frac{P(B|A)P(A)}{P(B)}$

   - For n events: $P(A_k|B)=\frac{P(B|A_k)P(A_k)}{\sum_{i=1}^{n}P(B|A_i)P(A_i)}$

7. **Total Probability Theorem:** $P(B)=\sum_{i=1}^{n}P(B|A_i)P(A_i)$.

8. **At Least and Exactly Probabilities:**

   - $P(\text{at least 2 events})=P(A\cap B)+P(A\cap C)+P(B\cap C)-2P(A\cap B\cap C)$

- $P(\text{exactly 1 event}) = P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(A \cap C) - 2P(B \cap C) + 3P(A \cap B \cap C)$

9. **Probability of exactly *k* successes in *n* trials:** $P(k) = \binom{n}{k} p^k (1-p)^{n-k}$.

10. **Expectation & Variance:** $E(X) = \sum xP(X = x)$, $Var(X) = E(X^2) - (E(X))^2$

Statistics:

The word **statistics** is derived from the Latin word 'status', which means a state. Statistics refers to the process of collecting, analysing, interpreting, presenting, and organising data. Statistics is used to analyse data by measuring central tendencies (mean, median, and mode) and dispersion (range, variance, and standard deviation). It plays a crucial role in various fields, including science, economics, and engineering.

Types of Averages

Averages refer to different measures of central tendency, including:

Mean:

The arithmetic average of a data set.

- **Discrete Data:** $\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$
- **Grouped Data:** $\bar{x} = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$ where f_i is the frequency and x_i is the mid-point.

Median:

The middle value when the data set is arranged in ascending order.

For Ungrouped Data:

- If the number of terms is odd: Median $=\frac{n+1}{2}$th term.
- If the number of terms is even: Median = Average of $\frac{n}{2}$ th and $\frac{n+1}{2}$ th term.

For Grouped Data:

Median $= l + \left(\frac{\frac{N}{2} - c}{f}\right) \times h$ where:

- $l =$ lower limit of the median class
- $f =$ frequency of median class
- $h =$ width of the median class
- $c =$ cumulative frequency of preceding median class

Mode:

Mode represents the most frequently occurring data point. For grouped data: $l + \left(\frac{f_1 - f_0}{2f_1 - f_0 - f_2}\right) \times h$ where:

- $l =$ lower boundary of the modal class
- $f_1 =$ frequency of modal class
- $f_0 =$ frequency before the modal class
- $f_2 =$ frequency after the modal class
- $h =$ class width

Dispersion in Statistics

Dispersion measures the spread of data points from the central value.

Absolute Measures of Dispersion

1. Range: $R = X_{\max} - X_{\min}$

2. Mean Deviation: $MD = \dfrac{\sum |x_i - \overline{x}|}{n}$

3. Variance: $\sigma^2 = \dfrac{\sum (x_i - \overline{x})^2}{n}$ or $\sigma^2 = \dfrac{\sum x_i^2}{n} - (\overline{x})^2$.

4. Standard Deviation: $\sigma = \sqrt{\sigma^2}$

Relative Measures of Dispersion

Used for comparing different data sets.

1. Coefficient of Range: $\dfrac{X_{\max} - X_{\min}}{X_{\max} + X_{\min}}$

2. Coefficient of Mean Deviation: $\dfrac{MD}{Mean}$ or $\dfrac{MD}{Median}$.

3. Coefficient of Variation (CV): $CV = \dfrac{\sigma}{\overline{x}} \times 100$.

Combined formulas:

1. Combined mean: $\overline{x} = \dfrac{n_1 \overline{x_1} + n_2 \overline{x_2}}{n_1 + n_2}$

2. Combined variance: $\sigma^2 = \dfrac{n_1 \sigma_1^2 + n_2 \sigma_2^2}{n_1 + n_2} + \dfrac{n_1 n_2}{(n_1 + n_2)^2} \left(\overline{X_1} - \overline{X_2}\right)^2$

Addition and Multiplication Rules

Addition of a Constant

- Mean: If a constant is added to each data point, the new mean is $\overline{x} + c$.
- Median: The new median will be the original median plus.
- Mode: The mode increases by c.

- Variance & Standard Deviation: These remain unchanged as dispersion does not change.
- Coefficient of Range: Unchanged under addition.
- Mean Deviation: Unchanged under addition.
- Coefficient of Variation (CV): Changes unpredictably under addition.

Multiplication by a Constant

- Mean: If each data point is multiplied by a constant k, the new mean is $k\bar{x}$.
- Median: The new median will be $k \times (\text{original median})$.
- Mode: The new mode will be $k \times (\text{original mode})$.
- Variance: The new variance will be $k^2 \times \sigma^2$.
- Standard Deviation: The new standard deviation will be $k \times \sigma$.
- Coefficient of Range: Unchanged under multiplication.
- Mean Deviation: Scales by $|k|$ under multiplication.
- Coefficient of Variation (CV): Unchanged under multiplication.

Example 1: A company has HR, IT, and Finance teams. HR has 15 males and 10 females, IT has 25 males and 15 females, and Finance has 30 males and 25 females. If a randomly chosen employee is female, what is the probability that they are from Finance?

Solution:

We need to find $P(F|Fe)$.
By the Baye's theorem,

$$P(F|Fe) = \frac{P(F)P(Fe|F)}{P(H)P(Fe|H) + P(I)P(Fe|I) + P(F)P(Fe|F)}$$

Since there are 3 sections, we have,

$$P(H)=\frac{1}{3}$$
$$P(I)=\frac{1}{3}$$
$$P(F)=\frac{1}{3}$$

From the given data,

$$P(Fe|H)=\frac{10}{25}$$
$$P(Fe|I)=\frac{15}{40}$$
$$P(Fe|F)=\frac{25}{55}$$

So, we have,

$$P(Q|M)=\frac{\frac{1}{3}\left(\frac{25}{55}\right)}{\frac{1}{3}\left(\frac{10}{25}\right)+\frac{1}{3}\left(\frac{15}{40}\right)+\frac{1}{3}\left(\frac{25}{55}\right)}$$
$$=\frac{\frac{25}{55}}{\frac{880+825+1000}{2200}}$$
$$=\frac{1000}{2705}$$
$$=\frac{200}{541}$$

Example 2: A box contains 3 fair coins and 2 biased coins (each biased coin lands on heads with probability 0.75). A coin is picked at random and flipped twice. If it lands heads both times, what is the probability that it was a biased coin?

Solution:

We need to find $P(B|HH)$.

From the given data, we have,

$$P(HH|B)=(0.75)^2$$
$$P(B)=\frac{2}{5}$$
$$P(HH)=P(HH|B)P(B)+P(HH|F)P(F)$$
$$=(0.75)^2\left(\frac{2}{5}\right)+\left(\frac{1}{2}\right)^2\left(\frac{3}{5}\right)$$

So, by Bayes' theorem, we have,

$$P(B|HH)=\frac{(0.75)^2\left(\frac{2}{5}\right)}{(0.75)^2\left(\frac{2}{5}\right)+\left(\frac{1}{2}\right)^2\left(\frac{3}{5}\right)}=\frac{0.225}{0.375}=\frac{3}{5}$$

Example 3: For an odd prime p, S_p is the set of all 2×2 matrices with only elements from the set $\{0,1,2,....,p-1\}$. Then, what is the probability that a randomly chosen matrix in S_p has determinant 0 given that the trace is divisible by p?

Solution:

Since the trace is divisible by p, for any non-zero element in the set $\{0,1,2,....,p-1\}$, the matrix is of the form $\begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$ or $\begin{bmatrix} a & b \\ c & p-a \end{bmatrix}$. The number of possible values for a are $p-1$.

For b and c, the number of possibilities in total is p^2. The total number of possibilities for a matrix such that the trace is divisible by p is p^3.

For the matrix $\begin{bmatrix} a & b \\ c & p-a \end{bmatrix}$ to have determinant 0, the matrices should be of the form $\begin{bmatrix} a & a \\ p-a & p-a \end{bmatrix}, \begin{bmatrix} a & p-a \\ a & p-a \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}, \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The total number of possibilities for the matrix to have a determinant 0 and that the trace is divisible by p is

$$\begin{aligned}((p-1)\times2)+2(p-1)+1&=2p-2+2p-2+1\\&=4p-3\end{aligned}$$

So, the probability that a randomly chosen matrix in S_p has determinant 0 given that the trace is divisible by p is $\frac{4p-3}{p^3}$.

Example 4: A four-digit number is chosen at random. What is the probability that its digits are in strictly increasing order?

Solution:

The total number of four-digit numbers is $9999-999=9000$.

Since the digits are in increasing order, we cannot have the digit 0 in the number. So, the digits can only be $\{1,2,3,4,5,6,7,8,9\}$.

Since the digits are strictly increasing, no number repetition is allowed, and only one possible combination is possible for one set of 4 digits, the number of possibilities of four-digit numbers with digits in strictly increasing order is

$$9c_4 = \frac{9\times8\times7\times6}{1\times2\times3\times4}$$
$$=126$$

The probability of selection is $\frac{126}{9000}=\frac{7}{500}$.

Example 5: Consider the set $S=\{2,3,5,7\}$. Let Q be the set of all 5-digit numbers formed using the elements from the set S. Then, what is the probability that a randomly chosen element from the set Q is divisible by 24 given that it is divisible by 4?

Solution:

For a number to be divisible by 4, the last 2 digits should be divisible by 4.

The possibilities for the digits are $32,52,72$. The number of possibilities for the first 3 digits are $4^3=64$.
The total number of possibilities for a number from the set to be divisible by 4 is $64\times3=192$

For a number to be divisible by 24, the number should be divisible by 3 and 8 since $\gcd(3,8)=1$

For a number to be divisible by 8, the last 3 digits should be divisible by 8.

The possibilities for the digits are $232,272,352,552,752$.

For a number to be divisible by 3, the sum of the digits must be divisible by 3.

The sum of the digits in each number are $7,11,10,12,14$. The remainders when divided by 3 are $1,2,1,0,2$.

The first 2 digits should be decided based on this.

Summing 2 digits and finding the remainder gives

$$\begin{aligned}
(2,2) &\to 1\\
(2,3) &\to 2\\
(2,5) &\to 1\\
(2,7) &\to 0\\
(3,3) &\to 0\\
(3,5) &\to 2\\
(3,7) &\to 1\\
(5,5) &\to 1\\
(5,7) &\to 0\\
(7,7) &\to 2
\end{aligned}$$

Mapping the 3-digit numbers based on the remainders, we have

$$\begin{aligned}
552 &\to (2,7),(3,3),(5,7)\\
232,352 &\to (2,3),(3,5),(7,7)\\
272,752 &\to (2,2),(2,5),(3,7),(5,5)
\end{aligned}$$

The number of possibilities for the number to be divisible by 24 is

$$\begin{aligned}
5+(2\times5)+(2\times6) &= 5+10+12\\
&= 27
\end{aligned}$$

The probability is $\frac{27}{192}=\frac{3}{64}$.

Example 6: A standard deck of 52 playing cards is shuffled, and one card is drawn randomly, with replacement, each time. What is the minimum number of draws required to ensure the probability of getting at least two kings is at least 0.05?

Solution:

Let n represent the number of draws.

The probability of getting a king is $p=\frac{1}{13}$.

The probability of not getting a king is $q=\frac{12}{13}$.

The probability of getting no kings in the n trials is $\binom{n}{0}\left(\frac{1}{13}\right)^0\left(\frac{12}{13}\right)^n$.

The probability of getting one king in the n trials is $\binom{n}{1}\left(\frac{1}{13}\right)^1\left(\frac{12}{13}\right)^{n-1}$.

The probability of getting at least two kings in the n trials is

$$1-\binom{n}{0}\left(\frac{1}{13}\right)^0\left(\frac{12}{13}\right)^n-\binom{n}{0}\left(\frac{1}{13}\right)^1\left(\frac{12}{13}\right)^{n-1}\geq 0.05$$

$$1-\frac{12^n}{13^n}-\frac{n(12)^{n-1}}{13^n}\geq 0.05$$

$$\frac{12^{n-1}}{13^n}(12+n)\leq 0.95$$

$$(0.9231)^n(12+n)\leq 11.4$$

For $n=3,(0.9231)^n(12+n)=11.7988$.

For $n=4,(0.9231)^n(12+n)=11.6175$.

For $n=5,(0.9231)^n(12+n)=11.3944$.

So, the minimum value of n is 5.

Example 7: Consider the region bounded by the curves $y=x^2-7x+6$ and $y=3x+6$. Then the probability that a point chosen in this region is below the line $y=0$ is

Solution:

The intersection of the curves can be calculated as

$$x^2-7x+6=3x+6$$
$$x^2-10x=0$$
$$x(x-10)=0$$
$$x=0,10$$

Also, the curve $y=x^2-7x+6$ intersects the $x-$axis at

$$x^2-7x+6=0$$
$$(x-1)(x-6)=0$$
$$x=1,6$$

The area bounded by the curves can be split as

$$A_1 = \left| \int_0^1 \left[3x+6-\left(x^2-7x+6\right) \right] dx \right|$$

$$A_2 = \left| \int_1^6 \left[0-\left(x^2-7x+6\right) \right] dx \right|$$

$$A_3 = \left| \int_1^6 \left[3x+6-0 \right] dx \right|$$

$$A_4 = \left| \int_6^{10} \left[3x+6-\left(x^2-7x+6\right) \right] dx \right|$$

The regions can be calculated as

1)

$$A_1 = \left[-\frac{x^3}{3}+10\left(\frac{x^2}{2}\right) \right]_0^1$$

$$= -\frac{1}{3}+5$$

$$= \frac{14}{3}$$

2)

$$A_2 = \left[-\frac{x^3}{3}+7\left(\frac{x^2}{2}\right)-6x \right]_1^6$$

$$= -72+126-36+\frac{1}{3}-\frac{7}{2}+6$$

$$= 24-\frac{19}{6}$$

$$= \frac{125}{6}$$

3)

$$A_3 = \left[3\left(\frac{x^2}{2}\right)+6x \right]_1^6$$

$$= 54+36-\frac{3}{2}-6$$

$$= \frac{165}{2}$$

4)

$$A_4=\left[-\frac{x^3}{3}+10\left(\frac{x^2}{2}\right)\right]_6^{10}$$
$$=-\frac{1000}{3}+500+72-180$$
$$=\frac{176}{3}$$

The probability can be calculated as

$$\frac{\frac{125}{6}}{\frac{14}{3}+\frac{125}{6}+\frac{165}{2}+\frac{176}{3}}=\frac{\frac{125}{6}}{\frac{28}{6}+\frac{125}{6}+\frac{495}{6}+\frac{352}{6}}$$
$$=\frac{\frac{125}{6}}{\frac{1000}{6}}$$
$$=\frac{\mathbf{1}}{\mathbf{8}}$$

Example 8: Choosing the numbers a,b from the natural numbers less than 115, what is the probability that a randomly chosen number 3^a+3^b is divisible by 10?

Solution:

The total number of possibilities for 3^a+3^b is 114×114.

The powers of 3 are $3,9,27,81,243,...$
For a number to be divisible by 10, the unit digit must be 0.

The powers can be classified as $3^{4n-3},3^{4n-2},3^{4n-1},3^{4n}$.

And, $114=4(29)-2$.

For one value of 3^{4n-3}, we have 28 pairs of 3^{4n-1}. And, there are 29 values of 3^{4n-3}.

For one value of 3^{4n-2}, we have 28 pairs of 3^{4n}. And, there are 29 values of 3^{4n}.

The number of possibilities of sum that is divisible by 10 is
$$(28\times29)+(28\times29)=2\times28\times29$$

So, the probability is

$$\frac{2\times28\times29}{114\times114}=\frac{406}{57\times57}$$
$$=\frac{406}{3249}$$

EXERCISE

1. For a discrete random variable, if $P(X=x)=\begin{cases}\frac{k}{x^2+4x+3}, & \text{for } x=0,1,2,....\\ 0, \text{ otherwise}\end{cases}$.

   Find the value of k.

2. Let $11=x_1>x_2>x_3>x_4>x_5$ be in an A.P. with a common difference d. If the standard deviation of the terms is $3\sqrt{2}$, then what is the value of x_4?

3. If $\sum_{i=1}^{15}(x_i-15)=15$ and $\sum_{i=1}^{15}(x_i-15)^2=255$, then what is the standard deviation for the items $(x_1,x_2,...,x_{15})$?

SOLUTIONS

1. Using the given data,

$$\sum_{x=0}^{\infty}\frac{k}{x^2+4x+3}=1$$

The series sum can be calculated as

$$\sum_{x=0}^{\infty}\frac{k}{x^2+4x+3}=\sum_{x=0}^{\infty}\frac{k}{(x+1)(x+3)}$$
$$=\sum_{x=0}^{\infty}\frac{k}{2}\left[\frac{(x+3)-(x+1)}{(x+1)(x+3)}\right]$$
$$=\frac{k}{2}\sum_{x=0}^{\infty}\left[\frac{1}{x+1}-\frac{1}{x+3}\right]$$
$$=\frac{k}{2}\left[\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+.....\right]$$

Solving further, we have,

$$\frac{k}{2}\left(1+\frac{1}{2}\right)=1$$
$$k=\frac{4}{3}$$

2. The given data can be rewritten as $11, 11+d, 11+2d, 11+3d, 11+4d$.

   Consider the terms $0, d, 2d, 3d, 4d$.
   The new mean is
   $$\frac{0+d+2d+3d+4d}{5}=2d$$

   Since variance and standard deviation are unchanged by the addition or subtraction of common terms, we have
   $$\frac{0^2+d^2+(2d)^2+(3d)^2+(4d)^2}{5}-(2d)^2=\left(3\sqrt{2}\right)^2$$
   $$\frac{30d^2}{5}-4d^2=18$$
   $$2d^2=18$$
   $$d=3$$

   Then, we have,
   $$x_4=11+3(3)=20$$

3. From the given data,
   $$\sum_{i=1}^{15}(x_i-15)=15$$
   $$\sum_{i=1}^{15}(x_i)-15\times15=15$$
   $$\sum_{i=1}^{15}(x_i)=240$$

   Then, the mean can be calculated as
   $$\bar{x}=\frac{240}{15}$$
   $$=16$$

   And,

$$\begin{aligned}\sum_{i=1}^{15}(x_i-\bar{x})^2 &= \sum_{i=1}^{15}(x_i-15-1)^2 \\ &= \sum_{i=1}^{15}(x_i-15)^2 - 2\sum_{i=1}^{15}(x_i-15) + \sum_{i=1}^{15}(1)^2 \\ &= 255-2(15)+15 \\ &= 240\end{aligned}$$

The standard deviation can be calculated as

$$\begin{aligned}\sigma &= \sqrt{\frac{\sum_{i=1}^{15}(x_i-\bar{x})^2}{15}} \\ &= \sqrt{\frac{240}{15}} \\ &= 4\end{aligned}$$