Newton School of Technology

NSAT

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NSAT Exam Guidelines

Exam Protocol

- Arrive at least 15 minutes early (by 2:45 PM) for exam setup and proctoring checks.
- All three 3 sections are mandatory and include a sectional cutoff that determines passing eligibility
- The exam will be proctored. Unfair means will lead to permanent disqualification.
- For MCQ-type Questions, Positive marking will be (+4) for correct answers, while negative marking (-1) is applicable for incorrect ones.
- There is no negative marking in coding section.

Joining Details & Device Restrictions

- Use a laptop/PC with screen-sharing and microphone access
- Dual-camera setup required:
    - Primary camera (webcam) facing your face
    - Secondary camera (mobile/external) showing your workspace
- Mobile phones are only allowed as a secondary camera (not for taking the test)
- Calculators are prohibited; use pen and paper for rough work
- Access the test portal at https://my.newtonschool.co/nsat/timeline
- Accept the calendar invite sent to you on test day as a reminder
- Maintain a stable internet connection to prevent interruptions during NSAT
- Keep both cameras ON throughout the test
- Position cameras properly:
    - Primary camera: face clearly visible
    - Secondary camera: desk, hands, and screen visible
- Ensure the secondary camera is placed steadily (not handheld)

Preparation

- Visit NSAT homepage to check compatibility before the exam.
- Use Google Chrome for optimal performance.
- Bring a pen and paper for rough work during the exam.
- Attempt a mock test to understand the exam pattern, for practice, and to confirm your PC compatibility.

Environment Considerations

- Sit in a quiet environment with no background noise to minimize distractions and avoid disqualification.
- Ensure the lighting in the room is appropriate for clear visibility and comfortable reading and writing.

CALCULUS

A real number l is called the **limit of the function** $f(x)$ as x tends to a if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $|x - a| < \delta$.
We write

$$\lim_{x \to a} f(x) = l$$

Left-Hand Limit:

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

Right-Hand Limit:

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

Existence of a Limit: A limit exists if and only if

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$$

Indeterminate Forms: A function may take an indeterminate form, such as

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty, 0^0, \infty^0, 1^\infty$$

Properties of Limits:

Property	Formula
Sum Rule	$\lim_{x \to a}(f(x)+g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
Product Rule	$\lim_{x \to a}(f(x) \cdot g(x)) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
Quotient Rule	$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

Some Standard Limits:

Function	Limit
$\lim_{x \to a} x^n$	a^n for $n \neq 0$
$\lim_{x \to a} \frac{x^n - a^n}{x - a}$	na^{n-1}
$\lim_{x \to 0} \frac{\sin x}{x}$	1
$\lim_{x \to 0} \frac{\tan x}{x}$	1
$\lim_{x \to 0} (1+x)^{\frac{1}{x}}$	e

L'Hopital's Rule: If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both results in an indeterminate form, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided $g'(x)\neq 0$.

Continuity: A function $f(x)$ is continuous at a point $x=a$ if

1. $f(a)$ is defined (i.e. $f(a)$ exists).
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x)=f(a)$.

If any of the above conditions fail, $f(x)$ is discontinuous at $x=a$.

Continuity on an Interval:

Open Interval (a,b): $f(x)$ is continuous on (a,b) if it is continuous at every point in (a,b).

Closed Interval $[a,b]$: $f(x)$ is continuous on $[a,b]$ if

- $f(x)$ is continuous on (a,b).
- $\lim_{x\to a^+} f(x)=f(a)$.
- $\lim_{x\to b^-} f(x)=f(b)$.

Derivative: The derivative of $f(x)$ at $x=a$ is given by

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

If this limit exists, $f(x)$ is differentiable at $x=a$.

Rule of Differentiation:

Rule	Formula
Constant Rule	$\frac{d}{dx}(c)=0$
Constant Multiplication	$\frac{d}{dx}(Ku)=K\frac{d}{dx}(u)$
Power Rule	$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$
Sum Rule	$\frac{d}{dx}(f(x)+g(x))=f'(x)+g'(x)$
Product Rule	$\frac{d}{dx}(uv)=u'v+uv'$
Quotient Rule	$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{u'v-uv'}{v^2}$

Chain Rule: If $y = f(u)$, where $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Common Derivatives:

Function	Derivative
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
a^x	$a^x \ln a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\cot x$	$-\csc^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\csc^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$

Leibniz Theorem: If u and v are differentiable n times, then

$$\frac{d^n}{dx^n}(uv) = \sum_{r=0}^{n} {}^nC_r u^r v^{n-r}$$

Partial Derivatives: For $z = f(x,y)$, the partial derivatives are

$$\frac{\partial z}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$\frac{\partial z}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Euler's Theorem on Homogeneous Functions: If $f(x,y)$ is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=nf(x,y)$$

Tangent: A tangent to a curve at a given point is a straight line that just touches the curve at that point without crossing it.

Normal: A normal is a line perpendicular to the tangent at the point of contact.

Equations of Tangent and Normal:
Let $y=f(x)$ be a given curve, and let $P(x_1,y_1)$ be a point on it.

Concept	Formula
Slope of Tangent	$f'(x_1)$
Slope of Normal	$-\frac{1}{f'(x_1)}$
Equation of Tangent	$y-y_1=f'(x_1)(x-x_1)$
Equation of Normal	$y-y_1=-\frac{1}{f'(x_1)}(x-x_1)$

Angle Between Two Curves: If two curves intersect at a point, the angle θ between them is given by

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

where m_1 and m_2 are the slopes of the tangents to the respective curves at the point of intersection.

Increasing and Decreasing Functions: A function $f(x)$ is said to be increasing or decreasing based on the sign of its derivative.

Condition	Nature of Function
$f'(x)>0$ for all x in (a,b)	$f(x)$ is strictly increasing in (a,b)
$f'(x)\geq 0$ for all x in (a,b)	$f(x)$ is monotonic increasing in (a,b)
$f'(x)<0$ for all x in (a,b)	$f(x)$ is strictly increasing in (a,b)
$f'(x)\leq 0$ for all x in (a,b)	$f(x)$ is monotonic increasing in (a,b)
$f'(x)=0$ for all x in (a,b)	$f(x)$ is a constant function

Approximation Using Differentiation: When a small change Δx occurs in x, the corresponding change in y is approximated as $\Delta y\approx dy=f'(x)\Delta x$

Approximate Value of a Function: If $y=f(x)$ then $f(a+\Delta x)\approx f(a)+f'(a)\Delta x$

First Derivative Test:

1. Find $f'(x)$ and solve $f'(x)=0$ for x.
2. Check the sign of $f'(x)$:
   - If $f'(x)$ changes from positive to negative, x is a local maximum.
   - If $f'(x)$ changes from negative to positive, x is a local minimum.
   - If no sign change occurs, it is neither a maxima nor minima (possibly an inflection point).

Second Derivative Test:

1. Find $f''(x)$.
2. Evaluate $f''(x)$ at critical points:
   - If $f''(x)>0$, x is a local minimum.
   - If $f''(x)<0$, x is a local minimum.
   - If $f''(x)=0$, test using the first derivative method.

Rolle's Theorem: If a function $f(x)$ satisfies

1. Continuity on $[a,b]$
2. Differentiability on (a,b)
3. $f(a)=f(b)$

Then, there exists at least one $c\in(a,b)$ such that

$$f'(c)=0$$

This means the function has a stationary point where the tangent is horizontal.

Lagrange's Mean Value Theorem: If $f(x)$ is

1. Continuous on $[a,b]$
2. Differentiable on (a,b)

Then, there exists $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

This states that the function has a point where the instantaneous rate of change equals the average rate of change.

Cauchy's Mean Value Theorem: If two functions $f(x)$ and $g(x)$ satisfy

1. Continuity on $[a,b]$
2. Differentiability on (a,b)
3. $g'(x)\neq 0$ in (a,b)

Then, there exists $c\in(a,b)$ such that

$$\frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

This compares the average rates of two functions.

Curvature of a Curve: Curvature K of a function $y=f(x)$ is

$$K=\frac{|f''(x)|}{(1+(f'(x))^2)^{\frac{3}{2}}}$$

This measures how sharply a curve bends at a point.

Envelope of a Family of Curves: For a family of curves $F(x,y,c)=0$, eliminating c from

$$F=0,\ \frac{dF}{dc}=0$$

gives the envelope, which is a curve touching all members of the family.

Indefinite Integrals: Indefinite integration is the process of finding a function $F(x)$ whose derivative is given by $f(x)$. It is represented as

$$\int f(x)dx=F(x)+C$$

where C is the constant of integration.

Properties of Indefinite Integrals:

Property	Formula
Linearity	$\int[f(x)\pm g(x)]dx=\int f(x)dx\pm\int g(x)dx$
Constant Multiplication	$\int af(x)dx=a\int f(x)dx$
Integration by Parts	$\int f(x)g(x)dx=f(x)\int g(x)dx-\int\left(\frac{d}{dx}f(x)\int g(x)dx\right)dx$

Common Integrals:

Function	Integral
x^n	$\frac{x^{n+1}}{n+1}+C,\ n\neq 1$
1	$x+C$
e^x	e^x+C
$\frac{1}{x}$	$\ln\lvert x\rvert+C$
a^x	$\frac{a^x}{\log a}+C$
$\sin x$	$-\cos x+C$
$\cos x$	$\sin x+C$
$\sec^2 x$	$\tan x+C$
$\sec x\tan x$	$\sec x+C$
$\csc x\cot x$	$-\csc x+C$
$\csc^2 x$	$-\cot x+C$
$\tan x$	$\ln\lvert\sec x\rvert+C$
$\cot x$	$\ln\lvert\sin x\rvert+C$
$\sec x$	$\ln\lvert\tan x+\sec x\rvert+C$
$\csc x$	$-\ln\lvert\csc x+\cot x\rvert+C$

$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1}x+C$
$\frac{-1}{\sqrt{1-x^2}}$	$\cos^{-1}x+C$
$\frac{1}{1+x^2}$	$\tan^{-1}x+C$
$\frac{-1}{1+x^2}$	$\cot^{-1}x+C$
$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$	$\sec^{-1}x+C$
$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$	$\csc^{-1}x+C$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\log\left\lvert\frac{x-a}{x+a}\right\rvert+C$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\log\left\lvert\frac{a+x}{a-x}\right\rvert+C$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}+C$
$\frac{1}{\sqrt{x^2-a^2}}$	$\log\left\lvert x+\sqrt{x^2-a^2}\right\rvert+C$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}+C$
$\frac{1}{\sqrt{x^2+a^2}}$	$\log\left\lvert x+\sqrt{x^2+a^2}\right\rvert+C$
$e^x[f(x)+f'(x)]$	$e^xf(x)+C$
$\sqrt{x^2-a^2}$	$\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\log\left\lvert x+\sqrt{x^2-a^2}\right\rvert+C$
$\sqrt{x^2+a^2}$	$\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left\lvert x+\sqrt{x^2+a^2}\right\rvert+C$
$\sqrt{a^2-x^2}$	$\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$

Integration of Rational Functions:

- When $\deg f(x)\geq\deg(g(x))$, use polynomial division to simplify.
- If the denominator is factored, use partial fractions.

Integration by Parts Rule:

$$\int uv\,dx=u\int v\,dx-\int\left(\frac{du}{dx}\int v\,dx\right)dx$$

Choose u using **ILATE** (Inverse, Logarithm, Algebraic, Trigonometric, Exponential).

Definite Integrals: A definite integral is an integral with limits of integration

$$\int_a^b f(x)dx = F(b) - F(a)$$

Properties of Definite Integrals:

Property	Formula
Reversal of Limits	$\int_a^b f(x)dx = -\int_b^a f(x)dx$
Additivity	$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
Even and Odd Functions	$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx, f(x) = even \\ 0, f(x) = odd \end{cases}$

Property-Based Integration:

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$\int_0^a f(x)dx = \frac{a}{2}\int_0^a f(x)dx \text{ ,if } f(2a-x) = f(x)$$

Leibniz Rule (Differentiation under the Integral Sign):

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x))\cdot h'(x) - f(g(x))\cdot g'(x)$$

Area Bounded by a Curve and the X-Axis: The area A enclosed by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$ is given by

$$A = \int_a^b f(x)dx$$

Area Bounded by a Curve and the Y-Axis: If the function is given as $x = f(y)$, the area between $y = c$ and $y = d$ is

$$A = \int_c^d f(y)dy$$

Area Between Two Curves: For two curves $y = f(x)$ and $y = g(x)$, where $g(x) \le f(x)$ in $[a, b]$, the enclosed area is

$$A=\int_a^b [f(x)-g(x)]dx$$

Similarly, if curves are given as $x=f(y)$ and $x=g(y)$ in $[c,d]$, the enclosed area is

$$A=\int_c^d [f(y)-g(y)]dy$$

Volume by Rotation about X-Axis: If the curve $y=f(x)$ is revolved around the x-axis from $x=a$ to $x=b$, the volume of the generated solid is

$$V=\pi\int_a^b [f(x)]^2 dx$$

Volume by Rotation about Y-Axis: For a function $y=f(x)$, rotated around the y-axis from $y=c$ to $y=d$, the volume of the generated solid is

$$V=\pi\int_c^d [f(y)]^2 dy$$

Shell Method: For functions revolving around the y-axis, another way to find volume is

$$V=2\pi\int_a^b xf(x)dx$$

Arc Length for Cartesian Functions: For a curve $y=f(x)$ from $x=a$ to $x=b$, the arc length is

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Arc Length for Parametric Equations: If a curve is given parametrically as $x=f(t)$, $y=g(t)$, the arc length is

$$L=\int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

Arc Length for Polar Coordinates: For a curve $r=f(\theta)$, the arc length is

$$L=\int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}\,d\theta$$

Surface Area by Rotation about the X-Axis: For $y=f(x)$, the surface area generated by revolving about the x-axis is

$$S=2\pi\int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$$

Surface Area by Rotation about the Y-Axis: For $x=f(y)$, the surface area generated by revolving around the y-axis is

$$S=2\pi\int_{c}^{d} f(y)\sqrt{1+(f'(y))^2}\,dy$$

Differential Equation: A differential equation is a mathematical equation that involves an independent variable, a dependent variable, and derivatives of the dependent variable with respect to the independent variable.
A differential equation can be classified into two main types:

Ordinary Differential Equation: A differential equation is called an ordinary differential equation if it contains derivatives with respect to a single independent variable.
For example, the equation

$$\frac{d^2y}{dx^2}+3\frac{dy}{dx}+5y=0$$

is an ordinary differential equation because it involves differentiation with respect to only one independent variable, which is x.

Partial Differential Equation: A differential equation is called a partial differential equation if it contains derivatives with respect to more than one independent variable.
For example, the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

is a partial differential equation because it involves differentiation with respect to two independent variables, which are x and y.

Order of a Differential Equation: The order of a differential equation is defined as the order of the highest order derivative present in the equation.
For example, in the equation

$$\frac{d^3y}{dx^3}+5\frac{dy}{dx}+y=0$$

the highest-order derivative is $\frac{d^3y}{dx^3}$, which means that the order of the differential equation is three.

Degree of a Differential Equation: The degree of a differential equation is the power of the highest-order derivative, provided the equation is expressed in polynomial form.
In the equation

$$\left(\frac{d^3y}{dx^3}\right)^2+5\frac{dy}{dx}+y=0$$

the highest-order derivative is raised to the power of two. Therefore, the degree of the differential equation is two.

Formation of a Differential Equation: A differential equation can be formed from a given function by eliminating arbitrary constants.

1. If the given function is of the form $f(x,y,a)=0$, then the differential equation is obtained by differentiating it with respect to x and eliminating the constant a.
2. If the given function is of the form $f(x,y,a,b)=0$, then the differential equation is obtained by differentiating it twice and eliminating the constants a and b.

Solutions of a Differential Equation: A solution of a differential equation is a function that satisfies the given equation. The solutions of differential equations can be classified into three types:

1. A **general solution** contains as many arbitrary constants as the order of the differential equation.
2. A **particular solution** is obtained by assigning specific values to the arbitrary constants in the general solution.
3. A **singular solution** is a solution that cannot be derived from the general solution.

First Order and First-Degree Differential Equations: A first-order and first-degree differential equation is an equation that involves only the first derivative of the dependent variable and does not contain any higher-order derivatives. The general form of such an equation is

$$\frac{dy}{dx}=f(x,y).$$

Variable Separable Method: If a differential equation can be expressed in the form

$$N(y)dy=M(x)dx$$

then the solution is obtained by integrating both sides separately.

Homogeneous Differential Equations: A differential equation of the form

$$\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$$

is said to be homogeneous if both $f(x,y)$ and $g(x,y)$ are homogeneous functions of the same degree. The solution is obtained by substituting $y=vx$ and reducing the equation into a separable form.

Linear Differential Equations: A linear first-order differential equation is of the form

$$\frac{dy}{dx}+P(x)y=Q(x)$$

The general solution of this equation is given by

$$y = e^{-\int P dx}\left[\int Q e^{\int P dx} dx + C\right]$$

Bernoulli's Equation: Bernoulli's equation is a nonlinear differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n,\ n \neq 1$$

By substituting $z = y^{1-n}$, the equation can be transformed into a linear equation and then solved accordingly.

Higher-Order Differential Equations: A second-order differential equation of the form

$$\frac{d^2y}{dx^2} = f(x)$$

can be solved by integrating both sides twice.

Solution Steps

1. Integrate $\frac{d^2y}{dx^2} = f(x)$ once to obtain $\frac{dy}{dx}$.
2. Integrate again to find y.

Exact Differential Equations: A first-order differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is called an exact differential equation if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Example 1: For positive integer n, define

$$f(n) = n + \frac{25+11n-4n^2}{5n+4n^2} + \frac{50+6n-4n^2}{10n+4n^2} + \ldots + \frac{41n-9n^2}{9n^2}.$$

Then, what is the value of $\lim_{n\to\infty} f(n)$?

Solution:

$$f(n) = n + \frac{25+11n-4n^2}{5n+4n^2} + \frac{50+6n-4n^2}{10n+4n^2} + \ldots + \frac{41n-9n^2}{9n^2}$$

$$= \left(\frac{25+11n-4n^2}{5n+4n^2}+1\right) + \left(\frac{50+6n-4n^2}{10n+4n^2}+1\right) + \ldots + \left(\frac{41n-9n^2}{9n^2}+1\right)$$

$$f(n) = \frac{16n+25}{5n+4n^2} + \frac{16n+50}{10n+4n^2} + \ldots + \frac{41n}{9n^2}$$

$$= \sum_{r=1}^{n} \frac{16n+25r}{5rn+4n^2} = \frac{1}{n}\sum_{r=1}^{n} \frac{16+25\left(\frac{r}{n}\right)}{5\left(\frac{r}{n}\right)+4}$$

$$\lim_{n\to\infty} f(n)=\int_0^1 \frac{16+25x}{5x+4}dx$$

$$=\int_0^1 \frac{(25x+20)-4}{5x+4}dx$$

$$=5-\frac{4}{5}\ln\frac{9}{4}$$

Example 2: For a positive integer n, let $b_n=\frac{1}{n}\left[\frac{(3n-1)!}{(2n-1)!}\right]^{\frac{1}{n}}$. Find $\lim_{n\to\infty} b_n$.

Solution:

$$b_n=\frac{1}{n}\left[\frac{(3n-1).....(3n-(n))(2n-1)!}{(2n-1)!}\right]^{\frac{1}{n}}$$

$$\Rightarrow b_n=\frac{1}{n}\left[n^n\left(3-\frac{1}{n}\right).....\left(3-\frac{n}{n}\right)\right]^{\frac{1}{n}}$$

$$\Rightarrow b_n=\left[\left(3-\frac{1}{n}\right).....\left(3-\frac{n}{n}\right)\right]^{\frac{1}{n}}$$

$$\text{Let}, L=\lim_{n\to\infty}\left[\left(3-\frac{1}{n}\right).....\left(3-\frac{n}{n}\right)\right]^{\frac{1}{n}}$$

$$\ln L=\lim_{n\to\infty}\frac{1}{n}\left[\ln\left(3-\frac{1}{n}\right)+\ln\left(3-\frac{2}{n}\right)+.....+\ln\left(3-\frac{n}{n}\right)\right].$$

$$\ln L=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(3-\frac{r}{n}\right)$$

$$\ln L=\int_0^1 \ln(3-x)dx$$

$$=-\left[(3-x)\ln(3-x)\right]_0^1+\int_0^1 d(3-x)$$

$$=-2\ln(2)+3\ln(3)+2-3$$

$$=\ln\left(\frac{27}{4e}\right)$$

$$L=\frac{27}{4e}$$

Example 3: Consider the functions $f:\left[0,\frac{\pi}{2}\right]\to[0,1]$ and $g:\left[0,\frac{\pi}{4}\right]\to[0,\infty)$ defined by $f(x)=\cos^2(2x)$ and $g(x)=\sqrt{\frac{\pi x}{4}-x^2}$. Find the value of $\frac{256}{\pi^3}\int_0^{\frac{\pi}{4}} f(x)g(x)dx$.

Solution:

Let $I=\int_0^{\frac{\pi}{4}} f(x)g(x)dx$. Then $I=\int_0^{\frac{\pi}{4}}\cos^2(2x)\sqrt{\frac{\pi x}{4}-x^2}dx$.

By King's rule,

$$I=\int_0^{\frac{\pi}{4}}\cos^2\left(2\left(\frac{\pi}{4}-x\right)\right)\sqrt{\frac{\pi}{4}\left(\frac{\pi}{4}-x\right)-\left(\frac{\pi}{4}-x\right)^2}dx$$

$$=\int_0^{\frac{\pi}{4}}\sin^2(2x)\sqrt{\frac{\pi^2}{16}-\frac{\pi}{4}x-\frac{\pi^2}{16}+\frac{\pi}{2}x-x^2}dx$$

$$=\int_0^{\frac{\pi}{4}}\sin^2(2x)\sqrt{\frac{\pi x}{4}-x^2}dx$$

So, $I=\frac{1}{2}\int_0^{\frac{\pi}{4}}\sqrt{\frac{\pi}{4}x-x^2}dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}\sqrt{\left(\frac{\pi}{8}\right)^2-\left(x-\frac{\pi}{8}\right)^2}dx$

Using $\int\sqrt{a^2-x^2}dx=\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\sin^{-1}\left(\frac{x}{a}\right)\right)+C$

$$\Rightarrow\frac{1}{2}\left[\left(x-\frac{\pi}{8}\right)\sqrt{\frac{\pi x}{4}-x^2}+\frac{\pi^2}{64}\sin^{-1}\left(\frac{x-\frac{\pi}{8}}{\frac{\pi}{8}}\right)\right]_0^{\frac{\pi}{4}}$$

$$\Rightarrow\frac{1}{2}\left[\left(0+\frac{\pi^3}{128}\right)-\left(0+\frac{-\pi^3}{128}\right)\right]=\frac{\pi^3}{128}$$

$$\frac{256}{\pi^3}\times\frac{\pi^3}{128}=2$$

Example 4: Let the function $f:[2,\infty)\to\mathbb{R}$ be defined by

$$f(t)=\begin{cases}(-1)^n 3 & \text{if } t=3n, n\in\mathbb{N}\\ \dfrac{(3n-t)}{3}f(3n-3)+\dfrac{(t-(3n-3))}{3}f(3n) & \text{if } 3n-3<t<3n, n\in\mathbb{N}\end{cases}.$$

Define $h(x)=\int_2^x f(t)dt$, $x\in(2,\infty)$. Let a denote the number of solutions of the equation $h(x)=0$ in the interval $(2,11]$ and $b=\lim_{x\to 2^+}\dfrac{b(x)}{x-2}$. Find the value of $10a-5b$.

Solution:

In the interval $(3n-3,3n)$, $f(t)=(-1)^{n-1}(6n-2t-3)$

$$g(x)=\begin{cases}\int_2^x(2t-3)dt & 2<x\le 3 & n=1\\ \int_2^3(2t-3)dt+\int_3^x(3-2t)dt & 3<x\le 6 & n=2\\ \int_2^3(2t-3)dt+\int_3^6(3-2t)dt+\int_6^x(2t-15)dt & 6<x\le 9 & n=3\\ \int_2^3(2t-3)dt+\int_3^x(3-2t)dt+\int_6^9(2t-3)dt+\int_9^x(21-2t)dt & 9<x\le 11 & n=4\end{cases}$$

$$g(x)=\begin{cases}x^2-3x+2 & 2<x\le 3 & n=1\\ 2-x^2+3x & 3<x\le 6 & n=2\\ x^2-15x+38 & 6<x\le 9 & n=3\\ -x^2+21x-124 & 9<x\le 11 & n=4\end{cases}$$

The solutions for $g(x)=0$, we have

$$x=\begin{cases}1,2 & 2<x\le 3 & n=1\\ \dfrac{3-\sqrt{17}}{2},\dfrac{3+\sqrt{17}}{2} & 3<x\le 6 & n=2\\ \dfrac{15-\sqrt{73}}{2},\dfrac{15+\sqrt{73}}{2} & 6<x\le 9 & n=3\\ \varnothing & 9<x\le 11 & n=4\end{cases}$$

There is only one solution in the interval.

$\therefore a=1$

$$b=\lim_{x\to 2^+}\frac{g(x)}{x-2}$$

Apply L'Hôpital's Rule

$$b=\frac{g'(2^+)}{1}=f(2^+)=1 \therefore 10a-5b=5$$

Example 5: Consider the functions $f,g:R\to R$ defined by $f(x)=-x^2+\frac{6}{5}$ and $g(x)=\begin{cases}3\left(1-\frac{6|x|}{5}\right),|x|\le\frac{5}{6}\\ 0,|x|>\frac{2}{3}\end{cases}$. Let γ be the area enclosed between the $x-$axis and the graph of $\min(f(x),g(x))$ for $|x|\le\frac{5}{6}$. Find γ.

Solution:

$f(x)=-x^2+\frac{6}{5}$

This represents a downward parabola.

$g(x)=3\left(1-\frac{6|x|}{5}\right)$

Both graphs have a vertex on the $y-$axis and the graphs are symmetric about that axis.

Intersection points of $f(x)$ and $g(x)$ in the first quadrant,

$f(x)=g(x)$

$-x^2+\frac{6}{5}=3\left(1-\frac{6x}{5}\right)$ [$|x|=x$ as in the first quadrant $x>0$]

$5x^2-6x+9=0$

$x=\frac{3}{5},3$

In the first quadrant $x=\frac{3}{5}$

For the first quadrant, in the interval $\left[0,\frac{3}{5}\right]$, $f(x)$ is the minimum.

In the interval $\left[\frac{3}{5},\frac{5}{6}\right]$, $g(x)$ is the minimum.

So, the area can be calculated as

$$\gamma=2\times\left[\int_0^{\frac{3}{5}}\left(-x^2+\frac{6}{5}\right)dx+\int_{\frac{3}{5}}^{\frac{5}{6}}\left(3\left(1-\frac{6x}{5}\right)\right)dx\right]=2\times\left[\frac{81}{125}+\frac{49}{500}\right]=\frac{373}{250}$$

Example 6: Let $p:[1,\infty)\to[0,\infty)$ be differentiable with $p(1)=3$ and $7\int_1^x p(t)dt = xp(x)-2x^2$, $x\in[1,\infty)$. Find $p(x)$.

Solution:

$7\int_1^x p(t)dt = xp(x)-2x^2, x\in[1,\infty)$

Differentiate both sides w.r.t. x

$$7(p(x)) = p(x)+xp'(x)-4x$$

$$xp'(x)-6p(x)=4x$$

$$p'(x)-\frac{6}{x}p(x)=4$$

The general form of a first order linear differential equation is

$y'(x)+P(x)y=Q(x)$

Here $P(x)=\frac{-6}{x}$, and $Q(x)=4$.

The integrating factor is obtained by exponentiating the integral of $p(x)$ with respect to x

$$e^{\int p(x)dx}=e^{\int\frac{-6}{x}dx}=e^{-6\ln|x|}=\frac{1}{x^6}$$

So, the solution of differential equation is

$$y\left(\frac{1}{x^6}\right)=\int(4)\frac{1}{x^6}dx$$

$$\frac{y}{x^6}=\frac{-1}{7x^7}+C$$

$$y=Cx^6-\frac{1}{7x}$$

$$p(x)=Cx^6-\frac{1}{7x}$$

Since $p(1)=3$, $C=\frac{22}{7}$

Then, $p(x)=\frac{22}{7}x^6-\frac{1}{7x}$.

Example 7: The curve C is given parametrically by:

$$x=t^3-3t,\ y=t^2-1$$

Find the equations of the two tangents at a double point (self-intersection).

Solution: A self-intersection occurs when two different values of t give the same (x, y).
Solving,

$$t_1^3 - 3t_1 = t_2^3 - 3t_2$$
$$t_1^2 - 1 = t_2^2 - 1$$

From the second equation,

$$t_1^2 = t_2^2 \Rightarrow t_1 = \pm t_2$$

Putting into the first equation,

$$t^3 - 3t = (-t)^3 - 3(-t)$$
$$t^3 - 3t = -t^3 + 3t$$
$$2t^3 = 6t$$
$$t(t^2 - 3) = 0$$

So $t = 0, \pm\sqrt{3}$.
The self-intersection occurs at (x,y) when $t = \sqrt{3}$ and $t = -\sqrt{3}$.
Evaluating

$$x = (\sqrt{3})^3 - 3(\sqrt{3}) = 3\sqrt{3} - 3\sqrt{3} = 0$$
$$y = (\sqrt{3})^2 - 1 = 2$$

The double point is $(0,2)$.
Differentiate x and y w.r.t. t,

$$\frac{dx}{dt} = 3t^2 - 3, \; \frac{dy}{dt} = 2t$$

Slope of the tangent at $t = \sqrt{3}$,

$$m_1 = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\sqrt{3}}{3(\sqrt{3})^2 - 3} = \frac{2\sqrt{3}}{9-3} = \frac{1}{\sqrt{3}}$$

Slope of the tangent at $t = -\sqrt{3}$,

$$m_2 = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sqrt{3}}{3(-\sqrt{3})^2 - 3} = \frac{-2\sqrt{3}}{9-3} = -\frac{1}{\sqrt{3}}$$

The two tangent lines at $(0,2)$ are

$$y - 2 = \frac{1}{\sqrt{3}}x \text{ and } y - 2 = -\frac{1}{\sqrt{3}}x$$

Example 8: Find the volume of the solid obtained by rotating the cardioid $r = 1 + \cos\theta$ about the x axis.

Solution:
Convert to Cartesian Form:

$$x = r\cos\theta = (1+\cos\theta)\cos\theta$$
$$y = r\sin\theta = (1+\cos\theta)\sin\theta$$

Use the standard volume formula in polar form:

$$V = \int_{\alpha}^{\beta} 2\pi y r d\theta$$
$$V = 2\pi \int_{0}^{\pi} (1+\cos\theta)^2 \sin\theta d\theta$$

Using substitution $u = \cos\theta \Rightarrow du = -\sin\theta d\theta$, this integral is evaluated

$$V = \frac{16\square}{3}$$

Example 9: Solve the differential equation:

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x},\ y(0)=1,\ y'(0)=0$$

Solution:
Solve the homogeneous equation:

$$r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0$$

So, $r = 2{,}2$,giving a solution:

$$y = (C_1 + C_2 x)e^{2x}$$

Find a particular solution: Since the RHS is e^{2x}:

$$y = Ax^2 e^{2x}$$

Plugging into the equation and solving for A, we get

$$A = \frac{1}{2}$$

So, $y = \frac{x^2}{2}e^{2x}$

The general solution is

$$y = (C_1 + C_2 x)e^{2x} + \frac{x^2}{2}e^{2x}$$

Apply initial conditions $y(0)=1,\ y'(0)=0$:

$$C_1 + 0 + 0 = 1 \Rightarrow C_1 = 1$$

Differentiating and solving for C_2, we get $C_2 = -2$.
The final solution is

$$y = (1-2x)e^{2x} + \frac{x^2}{2}e^{2x}$$

EXERCISE

1. If $\alpha = \lim_{x\to 0} \dfrac{\cos\left(x^2\right)-\left(1-x^4\right)^{\frac{1}{4}}+\left(\left(1+x^2\right)^{\frac{1}{3}}-1\right)\tan x}{x\tan^2(x)}$ then find the value of 9α.

2. Let the functions $f:(-1,1)\to R$ and $g:(-1,1)\to(-1,1)$ be defined by $f(x)=|2x-1|+|2x+1|$ and $g(x)=x-[x]$, where $[x]$ denotes the greatest integer less than or equal to x. Let $fog:(-1,1)\to R$ be the composite function defined by $(fog)(x)=f(g(x))$. Suppose c is the number of points in the interval $(-1,1)$ at which fog is NOT continuous, and suppose d is the number of points in the interval $(-1,1)$ at which fog is NOT differentiable. Find the value of $\dfrac{c}{d}$.

3. Find the value of the limit
$$\lim_{x\to\frac{\pi}{2}} \frac{\sin 3x+\sin x}{\left(2\sin 3x\sin\frac{x}{2}+\cos\frac{7x}{2}\right)-\left(\sqrt{2}+\sqrt{2}\cos 2x+\cos\frac{5x}{2}\right)}.$$

4. Integrate $\displaystyle\int \frac{4x^3+12x+8}{(x^2+1)^2(x+1)}dx$.

SOLUTIONS

1. $$\alpha = \lim_{x\to 0} \frac{\cos\left(x^2\right)-\left(1-x^4\right)^{\frac{1}{4}}+\left(\left(1+x^2\right)^{\frac{1}{3}}-1\right)\tan x}{x\tan^2(x)}$$

$$\alpha = \lim_{x\to 0} \frac{\left(1-\frac{x^4}{2}+\frac{x^8}{24}-\ldots\right)-\left(1-\frac{x^4}{4}+\frac{5x^8}{32}-\ldots\right)+\left(\left(1+\frac{1}{3}x^2-\frac{1}{9}x^4+\ldots\right)-1\right)\left(x+\frac{x^3}{3}+\frac{2}{15}x^5+\ldots\right)}{x.\frac{\tan^2 x}{x^2}.x^2}$$

$$=\lim_{x\to 0} \frac{\left(-\frac{1}{4}x^4+\ldots.\right)+\left(\frac{1}{3}x^3+\ldots.\right)}{x^3\left(\frac{\tan x}{x}\right)^2}$$

$$\alpha=\lim_{x\to 0}\frac{\frac{1}{3}-\frac{1}{4}x+.....}{\left(\frac{\tan x}{x}\right)^2}=\frac{\frac{1}{3}}{1}$$

$$9\alpha=9\times\frac{1}{3}=3$$

2. We know the composite function $(fog)(x)$ is discontinuous at the points, where $g(x)$ is discontinuous for given domain. And, since $g(x)$ is discontinuous at $x=0$ which lies in the interval $(-1,1)$, so value of $c=1$.

And, since $(fog)(x)$ is not differentiable at the point where $g(x)$ is not differentiable, as well as at those points also where $g(x)$ attains the values so that $f(g(x))$ is non-differentiable.
Since $g(x)$ is not continuous at $x=0\in(-1,1)$, so $fog(x)$ is not differentiable and as $f(x)=|2x-1|+|2x+1|$ is not differentiable at $x=-\frac{1}{2}$ and $\frac{1}{2}$, so $(fog)(x)$ is not differentiable for those x, for which $g(x)=-\frac{1}{2}$ or $\frac{1}{2}$.

But $g(x)\geq 0$, so $g(x)$ can be $\frac{1}{2}$ only and for $x=-\frac{1}{2}$ and $\frac{1}{2}$, $g(x)=\frac{1}{2}$.

So, $(fog)(x)$ is not differentiable at $x=-\frac{1}{2},0,\frac{1}{2}$, therefore value of $d=3$

$\therefore\frac{c}{d}=\frac{1}{3}$.

3. $$\lim_{x\to\frac{\pi}{2}}\frac{\sin 3x+\sin x}{\left(2\sin 3x\sin\frac{x}{2}+\cos\frac{7x}{2}\right)-\left(\sqrt{2}+\sqrt{2}\cos 2x+\cos\frac{5x}{2}\right)}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{2\sin 2x\cos x}{2\sin 3x\sin\frac{x}{2}+\left(\cos\frac{7x}{2}-\cos\frac{5x}{2}\right)-\sqrt{2}(1+\cos 2x)}$$

$$=\lim_{x\to\frac{\pi}{2}}\frac{2\sin 2x\cos x}{2\sin 3x\sin\frac{x}{2}-2\sin 3x\sin\frac{x}{2}-\sqrt{2}(2\cos^2 x)}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{2\sin 2x \cos x}{-2\sqrt{2}\cos^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{-(2\sin x \cos x)}{\sqrt{2}\cos x}$$

$$= \lim_{x \to \frac{\pi}{2}} \left(-\sqrt{2}\sin x\right)$$

$$= -\sqrt{2}$$

4. $$\int \frac{4x^3+12x+8}{(x^2+1)^2(x+1)}dx$$

$$= \int 4\left(\frac{x+1}{2(x^2+1)} + \frac{2}{(x^2+1)^2} - \frac{1}{2(x+1)}\right)dx$$

$$= 2\int \frac{x+1}{x^2+1}dx + 8\int \frac{1}{(x^2+1)^2}dx - 2\int \frac{1}{x+1}dx$$

$$= 2\int \frac{x}{x^2+1}dx + 2\int \frac{1}{x^2+1}dx + 8\int \frac{1}{(x^2+1)^2}dx - 2\int \frac{1}{x+1}dx$$

$$= \log(x^2+1) + 6\tan^{-1}x + \frac{4x}{x^2+1} - 2\log(x+1) + C$$